Suggested Exercise 11

1. From HW7, Since Σ is compact, $\exists\, p \in S$ s.t. $K(p) > 0$

By Gauss-Bonnet's Theorem,

$$\int_\Sigma K \, dA = 2\pi \chi(\Sigma) \leq 0 \quad \text{as } \Sigma \text{ is not homeomorphic to sphere.}$$

So $\exists\, p_1 \in \Sigma$ s.t. $K(p_1) < 0$

Since Gauss curvature is continuous, $\exists\, p_2 \in \Sigma$ s.t $K(p_2) = 0$

2. $f(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u) \qquad 0<u<2\pi, \ 0<v<2\pi$

$$f_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$$

$$f_v = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0)$$

$$N = \frac{f_u \times f_v}{|f_u \times f_v|} = (-\cos u \cos v, -\cos u \sin v, -\sin u)$$

$$f_{uu} = (-\cos u \cos v, -\cos u \sin v, -\sin u)$$

$$f_{uv} = (\sin u \sin v, -\sin u \cos v, 0)$$

$$f_{vv} = (-(2+\cos u)\cos v, -(2+\cos u)\sin v, 0)$$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (2+\cos u)^2 \end{pmatrix} \qquad (h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (2+\cos u)\cos u \end{pmatrix}$$

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{(2+\cos u)\cos u}{(2+\cos u)^2} = \frac{\cos u}{2+\cos u}$$

$$\int_T K \, dA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{2+\cos u} \sqrt{\det(g_{ij})} \, du\,dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos u \, du\,dv = 0$$

3a. Since α is closed, $\exists\, s_0 \in [0,L]$ s.t. $|\alpha(s_0)| = \max_{s\in[0,L]} |\alpha(s)|$

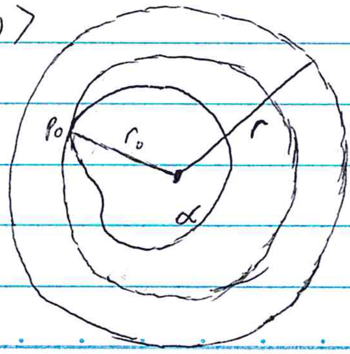
$$0 = \frac{d}{ds} |\alpha(s)|^2 \Big|_{s=s_0} = 2\langle \alpha'(s_0), \alpha(s_0)\rangle \implies \alpha(s_0) \,//\, N(s_0)$$

$$0 \geq \frac{d^2}{ds^2} |\alpha(s)|^2 \Big|_{s=s_0} = 2\langle \alpha''(s_0), \alpha(s_0)\rangle + 2\langle \alpha'(s_0), \alpha'(s_0)\rangle$$

$$-\langle \alpha''(s_0), \alpha(s_0)\rangle \geq 1$$

$$|\alpha(s_0)|\,|k(s_0)| \geq 1$$

$$|k(s_0)| \geq \frac{1}{|\alpha(s_0)|} \geq \frac{1}{r}$$



k at $p_0 \geq \frac{1}{r_0} \geq \frac{1}{r}$

3b. From HW 11, $2A \le \left(\int_0^L |\alpha(s)|^2\right)^{\frac{1}{2}} L^{\frac{1}{2}}$

$$\le \left(\int_0^L r^2\right)^{\frac{1}{2}} L^{\frac{1}{2}}$$

$$= rL$$

$$A \le \frac{rL}{2}$$

"=" holds if and only if $|\alpha(s)| = r$ (i.e. α is a circle of radius r)

4a. $\beta(s) = \alpha(s) - rN(s)$

$$\beta'(s) = T(s) - r(-k(s)T(s))$$

$$= (1 + rk(s))T(s)$$

$$\text{Length}(\beta) = \int_0^L |\beta'(s)|\, ds = \int_0^L 1 + rk(s)\, ds$$

$$= L + r\int_0^L k(s)\, ds$$

$$= \text{Length}(\alpha) + 2\pi r$$

4b. Let $T_\beta(s)$ be tangent of β, $N_\beta(s)$ be normal of β

Since $\beta'(s) \parallel T(s)$, $T(s) = T_\beta(s)$ and $N(s) = N_\beta(s)$

$\text{Area}(\Omega_\beta) = \frac{1}{2}\int_{\partial\Omega_\beta} \langle F, n\rangle$ where $F(x,y) = (x,y)$ n: outward normal to $\partial\Omega_\beta$

$$= \frac{1}{2}\int_0^L \langle \beta(s), n(s)\rangle\, |\beta'(s)|\, ds$$

$$= \frac{1}{2}\int_0^L \langle \alpha(s) - rN(s), -N(s)\rangle\, (1 + rk(s))\, ds$$

$$= \frac{1}{2}\int_0^L \langle \alpha(s), -N(s)\rangle ds + \frac{r}{2}\int_0^L k(s)\langle \alpha(s), -N(s)\rangle ds + \frac{1}{2}\int_0^L r(1 + rk(s))\, ds$$

$$= \text{Area}(\Omega_\alpha) + \frac{r}{2}\int_0^L k(s)\langle \alpha(s), -N(s)\rangle\, ds + \frac{rL}{2} + \frac{r^2}{2}\int_0^L k(s)\, ds$$

$$= \text{Area}(\Omega_\alpha) + \frac{r}{2}\int_0^L k(s)\langle \alpha(s), -N(s)\rangle\, ds + \frac{rL}{2} + \pi r^2$$

$$\int_0^L k(s)\langle \alpha(s), -N(s)\rangle ds = \int_0^L \langle \alpha(s), -k(s)N(s)\rangle\, ds$$

$$= \int_0^L \langle \alpha(s), \tfrac{d}{ds}(-T(s))\rangle\, ds$$

$$= \int_0^L \underbrace{\tfrac{d}{ds}\langle \alpha(s), -T(s)\rangle}_{0} + \langle T(s), T(s)\rangle\, ds = L$$

$$\text{Area}(\Omega_\beta) = \text{Area}(\Omega_\alpha) + rL + \pi r^2$$